Partitions and A Multi-dimensional Continued Fraction Algorithm

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Goal

Use the dynamics of the triangle map (a type of multi-dimensional continued fraction algorithm) to create an almost internal symmetry on the space of all partitions of a given integer N.

Outline

2 dimensional case.
For motivation

1. Partitions
2. The Farey Tree, Farey map and its links to partitions
3. The Triangle Map and its link to partitions
4. Method to Generate Many New Partition Identities
5. Why the triangle map? Questions.

Partitions

$p(n)$ is the number of ways of writing n as the sum of less than or equal t positive integers (ordering not mattering).
$p(7) = 15$ since

$$\begin{array}{ccc} 7 & 6+1 & 5+2 \\ 5+1+1 & 4+3 & 4+2+1 \\ 4+1+1+1 & 3+3+1 & 3+2+2 \\ 3+2+1+1 & 3+1+1+1+1 & 2+2+2+1 \\ 2+2+1+1+1 & 2+1+1+1+1+1 & 1+1+1+1+1+1+1. \end{array}$$

Or

as

$$\begin{array}{cccc} (7) & (6,1) & (5,2) & (5,1^2) \\ (4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\ (3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\ (2^2,1^3) & (2,1^5) & (1^7). & \end{array}$$

Partitions

$$
\begin{array}{cccc}
(7) & (6,1) & (5,2) & (5,1^2) \\
(4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\
(3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\
(2^2,1^3) & (2,1^5) & (1^7). &
\end{array}
$$

or as

$$
\begin{array}{cccc}
(7) \times [1] & (6,1) \times [1,1] & (5,2) \times [1,1] & (5,1) \times [1,2] \\
(4,3) \times [1,1] & (4,2,1) \times [1,1,1] & (4,1) \times [1,3] & (3,1) \times [2,1] \\
(3,2) \times [1,2] & (3,2,1) \times [1,1,2] & (3,1) \times [1,4] & (2,1) \times [3,1] \\
(2,1) \times [2,3] & (2,1) \times [1,5] & (1) \times [7]. &
\end{array}
$$

Partitions

The multiplicities

The parts

$$\lambda = (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \vdash N$$

dimension m

means

$$N = k_1\lambda_1 + \ldots + k_m\lambda_m.$$

$$= (k_1, \ldots, k_m) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$$

Partitions

There are many remarkable identities.

For example, Andrew and Eriksson's *Integer Paritions* starts with discussing Euler's identity:

"*Every number has as many integer partitions into odd parts as into distinct parts.*"

Rogers-Ramanujan
Identities

Partitions

Two Questions

1. How to find possible identities
2. How to prove them

Goal: use a dynamical system to generate many new identities. The proofs will be actually straightforward

Partitions

To a given partition

$$(\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m]$$

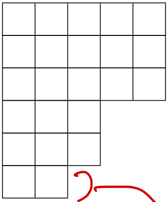
we associate the *Young shape*, a diagram $k_1 + \cdots + k_m$ rows such that there are k_1 rows with λ_1 squares on top of k_2 rows with λ_2 squares, and so on.

Partitions

For example, the Young shape for

$$(5, 3, 2) \times [3, 2, 1] \vdash 23$$

is



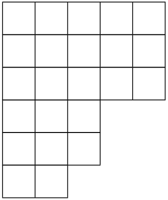
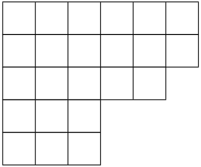
3 rows with 5 boxes

2 rows of 3 boxes

one row of 2 boxes

Partitions

Flip a Young shape, turning the rows into columns, to get the *conjugate partition*
Flipping the Young shape of the partition $(5,3,2) \times [3,2,1] \vdash 23$ of the previous example gives us the Young shape


$\sim_{\mathcal{C}}$


which represents the conjugate partition

$$(5,3,2) \times [3,2,1] \sim_{\mathcal{C}} (6,5,3) \times [2,1,21]$$

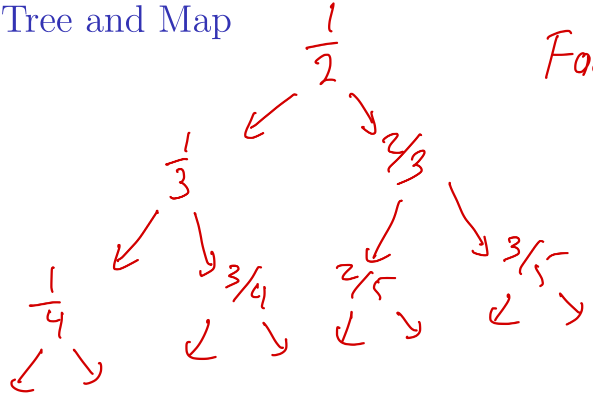
Partitions

$$(\lambda_1, \lambda_2) \times [k_1, k_2] \sim_{\mathcal{C}} (k_1 + k_2, k_1) \times [\lambda_2, \lambda_1 - \lambda_2]$$

and in general

$$\begin{gathered}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \sim_{\mathcal{C}} \\ (k_1 + \ldots + k_m, k_1 + \ldots + k_{m-1}, \ldots, k_1) \\ \times \\ [\lambda_m, \lambda_{m-1} - \lambda_m, \ldots, \lambda_1 - \lambda_2]\end{gathered}$$

Farey Tree and Map



Farey Tree

Every rational number in $(0,1)$ will eventually appear.

Farey Tree and Map $\quad 0 < \lambda_2 < \lambda_1$

$$\frac{\lambda_2}{\lambda_1} \sim \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{array}{ccc} & \dfrac{\lambda_2}{\lambda_1} & \\ \swarrow & & \searrow \\ \dfrac{\lambda_2}{\lambda_1+\lambda_2} & & \dfrac{\lambda_1}{\lambda_1+\lambda_2} \end{array}$$

Metrics

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1+\lambda_2 \\ \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1+\lambda_2 \\ \lambda_1 \end{pmatrix}$$

Farey Tree and Map

Farey tree again

$$
\begin{array}{ccccccc}
 & & & (2,1) & & & \\
 & \swarrow & & & & \searrow & \\
 & (3,1) & & & & (3,2) & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
(4,1) & & (4,3) & & (5,2) & & (5,3)
\end{array}
$$

$$
\begin{array}{ccc}
 & (\lambda_1, \lambda_2) & \\
\swarrow & & \searrow \\
(\lambda_1+\lambda_2, \lambda_2) & & (\lambda_1+\lambda_2, \lambda_1)
\end{array}
$$

Farey Tree and Map

Inverse

If $(\lambda_1, \lambda_2) \to (\lambda_1 + \lambda_2, \lambda_2)$

the inverse is

$(\mu_1, \mu_2) \to (\mu_1 - \mu_2, \mu_2)$

If $(\lambda_1, \lambda_2) \to (\lambda_1 + \lambda_2, \lambda_1)$

the inverse is

$(\mu_1, \mu_2) \to (\mu_2, \mu_1 - \mu_2)$

Farey Tree and Map

The inverse map

$$(\lambda_1, \lambda_2) \begin{array}{lll} \xrightarrow{F_0} & (\lambda_2, \lambda_1 - \lambda_2) & \text{if} \quad \lambda_1 < 2\lambda_2 \\ \xrightarrow{F_1} & (\lambda_1 - \lambda_2, \lambda_2) & \text{if} \quad \lambda_1 > 2\lambda_2 \end{array}$$

Via matrices

$$F_0 \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 - \lambda_2 \end{pmatrix}$$

$$F_1 \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_2 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 - \lambda_2 \end{pmatrix}$$

Farey Tree and Map

This gives us paths:

$$(7,4) \xrightarrow{F_0} (4,3) \xrightarrow{F_0} (3,1) \xrightarrow{F_1} (2,1).$$

How to get partitions:

$$\begin{aligned}
(7,4)\times[k_1,k_2] &\xrightarrow{\tilde{F_0}} (4,3)\times[k_1+k_2,k_1]\\
&\xrightarrow{\tilde{F_0}} (3,1)\times[2k_1+k_2,k_1+k_2]\\
&\xrightarrow{\tilde{F_1}} (2,1)\times[2k_1+k_2,3k_1+2k_2]
\end{aligned}$$

All partition the same number

$7k_1 + 4k_2$

Farey Tree and Map

The extended Farey map:

$$(\lambda_1, \lambda_2)\times[k_1, k_2] \begin{array}{l} \xrightarrow{\tilde{F}_0} \quad (\lambda_2, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_1] \quad \text{if} \quad \lambda_1 < 2\lambda_2 \\ \xrightarrow{\tilde{F}_1} \quad (\lambda_1 - \lambda_2, \lambda_2) \times [k_1, k_1 + k_2] \quad \text{if} \quad \lambda_1 > 2\lambda_2 \end{array}$$

In dynamics, this is called the natural extension

Farey Tree and Map

Via matrices

$$
\begin{aligned}
\tilde{F}_0 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} F_0 & 0 \\ 0 & (F_0^{-1})^T \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k_1 \\ k_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k_1 \\ k_2 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_2 \\ \lambda_1 - \lambda_2 \\ k_1 + k_2 \\ k_1 \end{pmatrix}
\end{aligned}
$$

must have entries ≥ 0,

or multiplicities could become negative

Farey Tree and Map

$$
\tilde{F}_1 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & (F_1^{-1})^T \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k_1 \\ k_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k_1 \\ k_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} \lambda_1 - \lambda_2 \\ \lambda_2 \\ k_1 \\ k_1 + k_2 \end{pmatrix}
$$

must have entries ≥ 0

Farey Tree and Map

Paths:

$$
\begin{aligned}
\underline{(19,8)\times[1,0]} &\xrightarrow{\tilde{F}_1} (11,8)\times[1,1]\\
&\xrightarrow{\tilde{F}_0} (8,3)\times[2,1]\\
&\xrightarrow{\tilde{F}_1} (5,3)\times[2,3]\\
&\xrightarrow{\tilde{F}_0} (3,2)\times[5,2]\\
&\xrightarrow{\tilde{F}_0} (2,1)\times[7,5]
\end{aligned}
$$

All are partitions of 19.

Farey Tree and Map

Respects conjugation:
The diagram

Seems to be important.
Rarely happens in most generalization

$$\begin{array}{rcl}
(\lambda_1,\lambda_2)\times[k_1,k_2] & \sim_{\mathcal{C}} & (k_1+k_2,k_1)\times[\lambda_2,\lambda_1-\lambda_2] \\
\tilde{F}_0\downarrow & & \uparrow\tilde{F}_0 \\
(\lambda_2,\lambda_1-\lambda_2)\times[k_1+k_2,k_1] & \sim_{\mathcal{C}} & (2k_1+k_2,k_1+k_2)\times \\
 & & [\lambda_1-\lambda_2, 2\lambda_2-\lambda_1]
\end{array}$$

when $\lambda_2 \geq \lambda_1 - \lambda_2$, and the diagram

$$\begin{array}{rcl}
(\lambda_1,\lambda_2)\times[k_1,k_2] & \sim_{\mathcal{C}} & (k_1+k_2,k_1)\times[\lambda_2,\lambda_1-\lambda_2] \\
\tilde{F}_1\downarrow & & \uparrow\tilde{F}_1 \\
(\lambda_1-\lambda_2,\lambda_2)\times[k_1,k_1+k_2] & \sim_{\mathcal{C}} & (2k_1+k_2,k_1)\times[\lambda_2,\lambda_1-2\lambda_2]
\end{array}$$

when $\lambda_2 \leq \lambda_1 - \lambda_2$, are both commutative.

Farey Tree and Map

Theorem

Let $n \geq 2$ be an integer. Every partition of n can be obtained from the dynamics of the extended Farey map $\tilde{F}$.

Theorem

Let $n \geq 2$ be an integer.

$$p(2,n) = \frac{1}{2}\sum_{r=1}^{n-1}\left(\underbrace{depth\left(\frac{r}{n}\right)}_{\text{w.r.t. Farey tree}} - 1\right)\overbrace{\sigma_0((r,n))}^{\text{\# of divisors of gcd}(r,n)}.$$

(Different from Kim (2012).)

(Quite Different)

The Triangle Map

Number-theoreti

A dynamical system on simplices.

Earlier work

(TG) (2001)

S. Assaf, L. Chen, T. Cheslack-Postava, B. Cooper, A. Diesl, TG, M. Lepinski and A. Schuyler (2005)

A. Messaoudi, A. Nogueira, and F. Schweiger (2009)

V. Berthé, W. Steiner and J. Thuswaldner (2021)

Fougeron and A. Skripchenko (2021)

C.Bonanno, A. Del Vigna and S. Munday (2021)

C. Bonanno and A. Del Vigna (2021)

H. Ito (2023)

Dynamical

papers

The Triangle Map

Many

Roots of Multi-dimensional Continued Fractions:

1. Generalize the fact that a number has an eventually periodic continued fraction expansion if and only if it is a quadratic irrational.
2. Finding best Diophantine approximations of n-tuples of reals by n-tuples of rationals
3. As a rich source of dynamical systems, starting with Gauss on continued fractions all the way to the current work on interval exchange maps.

The Triangle Map

Farey map as iterative system

$F:(0,1)\to(0,1)$

$$F(x)=\begin{cases}\dfrac{1-x}{x}, & 1/2<x<1\\[2mm] \dfrac{x}{1-x}, & 0<x<1/2\end{cases}$$

or

$$(1,x)\begin{cases}\to (x,1-x), & 1/2<x<1\\ \to (1-x,x), & 0<x<1/2\end{cases}$$

Iterate to get continued fraction expansion

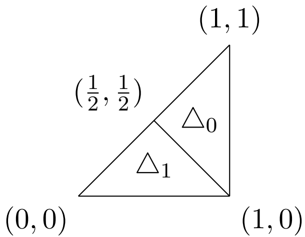
The Triangle Map

Set

$$\begin{aligned}
\triangle &:= \{(x_1,\ldots,x_n)\in\mathbb{R}^n : 1 > x_1 > \cdots > x_n > 0\}\\
\triangle_0 &:= \{x_1,\ldots,x_n)\in\triangle : x_1 + x_n > 1\}\\
\triangle_1 &:= \{x_1,\ldots,x_n)\in\triangle : x_1 + x_n > 1\}\\
\triangle_D &:= \{x_1,\ldots,x_n)\in\triangle : x_1 + x_n = 1\}
\end{aligned}$$

in dynamics often ignored, as is a set of measure 0.

When $n = 2$, we have



The Triangle Map

The slow-Triangle map $T : \triangle_0 \cup \triangle_1 \to \triangle$ is

$$T(x_1, \ldots, x_n) = \begin{cases} T_0(x_1, \ldots, x_n), & \text{if } x_1 + x_n > 1 \\ T_1(x_1, \ldots, x_n), & \text{if } x_1 + x_n < 1 \end{cases}$$

$$= \begin{cases} \left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}, \frac{1-x_1}{x_1}\right), & \text{if } x_1 + x_n > 1 \\ \left(\frac{x_1}{1-x_n}, \ldots, \frac{x_n}{1-x_n}\right), & \text{if } x_1 + x_n < 1 \end{cases}$$

Clear denominators

(Pass to projective space)

The Triangle Map

$$\begin{aligned}
\triangle &:= \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 > x_1 > \cdots > x_n > 0\} \\
\triangle_0 &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n > x_0\} \\
\triangle_1 &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n < x_0\} \\
\triangle_D &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n = x_0\}
\end{aligned}$$

and define the slow-Triangle map $T : \triangle_0 \cup \triangle_1 \to \triangle$ by

$$\begin{aligned}
T(x_0, \ldots, x_n) &= \begin{cases} T_0(x_0, \ldots, x_n), & \text{if } x_1 + x_n > x_0 \\ T_1(x_0, \ldots, x_n), & \text{if } x_1 + x_n < x_0 \end{cases} \\
&= \begin{cases} (x_1, x_2, \ldots, x_n, x_0 - x_1), & \text{if } x_1 + x_n > x_0 \\ (x_0 - x_n, x_1, x_2, \ldots, x_n), & \text{if } x_1 + x_n < x_0 \end{cases}
\end{aligned}$$

The Triangle Map

$$(7,4,2) \xrightarrow{T_1} (6,4,2) \quad \text{since } 7 > 4+2$$

$$(7,5,4) \xrightarrow{T_0} (5,4,2\quad), \quad \text{since } 7 < 5+4$$

The Triangle Map

$$T\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{cases} T_0\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, & \text{if } x_1 + x_n > x_0 \\ T_1\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, & \text{if } x_1 + x_n < x_0 \end{cases}$$

The Triangle Map

where

$$T_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Thus for $n = 2$, we have

$$T_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Triangle Map

$$T_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad T_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

All entries ≥ 0.

This will be important

The Triangle Map

The *extended slow-Triangle map* $\tilde{T}$ will act on

$$(\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m] = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \\ k_1 \\ \vdots \\ k_m \end{pmatrix}$$

as the action of two $2m \times 2m$ matrices on column vectors in $\mathbb{R}^{2m}$, with the matrices

$$\begin{pmatrix} T_0 & 0 \\ 0 & (T_0^{-1})^\top \end{pmatrix}, \begin{pmatrix} T_1 & 0 \\ 0 & (T_1^{-1})^\top \end{pmatrix}.$$

non zero entries

The Triangle Map

$$\begin{array}{c}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_0 \downarrow \\ (\lambda_2, \lambda_3 \ldots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \ldots, k_m, k_1]\end{array}$$

if $\lambda_2 + \lambda_m > \lambda_1$ and

$$\begin{array}{c}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_1 \downarrow \\ (\lambda_1 - \lambda_m, \lambda_2 \ldots, \lambda_m,) \times [k_1, \ldots, k_{m-1}, k_1 + k_m]\end{array}$$

if $\lambda_2 + \lambda_m < \lambda_1$

The Triangle Map

A path

$$\begin{aligned}
(14,7,6,5)\times[1,0,0,0] &\xrightarrow{\tilde{T}_1} (9,7,6,5)\times[1,0,0,1]\\
&\xrightarrow{\tilde{T}_0} (7,6,5,2)\times[1,0,1,1]\\
&\xrightarrow{\tilde{T}_0} (6,5,2,1)\times[1,1,1,1]
\end{aligned}$$

as $6 = 5 + 1$

$(\lambda_1 = \lambda_2 + \lambda_4)$

must, for now, stop

The Triangle Map

Respects conjugation:

Theorem

The diagram

Again, rarely happens for other multidimensional continued fractions

$$\begin{array}{ccc} (\bar{\lambda}) \times [\bar{k}] & \sim_C & \tilde{T}_0((\bar{\mu}) \times [\bar{l}]) \\ \tilde{T}_0 \downarrow & & \uparrow \tilde{T}_0 \\ \tilde{T}_0((\bar{\lambda} \times [\bar{k}])) & \sim_C & (\bar{\mu}) \times [\bar{l}] \end{array}$$

when $\lambda_2 + \lambda_m > n_1$ *and*

$$\begin{array}{ccc} (\bar{\lambda}) \times [\bar{k}] & \sim_C & \tilde{T}_0 1((\bar{\mu}) \times [\bar{l}]) \\ \tilde{T}_1 \downarrow & & \uparrow \tilde{T}_1 \\ \tilde{T}_1((\bar{\lambda} \times [\bar{k}])) & \sim_C & (\bar{\mu}) \times [\bar{l}] \end{array}$$

when $\lambda_2 + \lambda_m < \lambda_1$ *are both commutative.*

The Triangle Map

What if

$$\lambda_1 = \lambda_2 + \lambda_m$$

dim m

$$\begin{array}{c} (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \tilde{T}_D \downarrow \\ (\lambda_2, \lambda_3 \ldots, \lambda_m) \times [k_1 + k_2, k_3, \ldots, k_1 + k_m] \end{array}$$

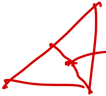
dim m-1

Drop dimension

The Triangle Map

$\tilde{T}_D$ actually quite natural

$(3,2,1) \times [4,3,5]$



$(3,2,1)$

$(3,2,1)$ in both Δ_0 and Δ_1

$$\begin{tikzcd}
 & (2,1,1) \times [7,5,4] \arrow[dr, "\text{concatenate}"] & \\
(3,2,1) \times [4,3,5] \arrow[ur, "\tilde{T}_0"] \arrow[rr, "\tilde{T}_D"'] \arrow[dr, "\tilde{T}_1"'] & & (2,1) \times [7,9] \\
 & (2,2,1) \times [4,3,9] \arrow[ur, "\text{concatenate}"'] &
\end{tikzcd}$$

The Triangle Map

$$
\begin{aligned}
(14,7,6,5) &\xrightarrow{\tilde{T}_1} (9,7,6,5)\times[1,0,0,1]\\
&\xrightarrow{\tilde{T}_0} (7,6,5,2)\times[1,0,1,1]\\
&\xrightarrow{\tilde{T}_0} (6,5,2,1)\times[1,1,1,1]\\
&\xrightarrow{\tilde{T}_D} (5,2,1)\times[2,1,2]\\
&\xrightarrow{\tilde{T}_1} (4,2,1)\times[2,1,4]\\
&\xrightarrow{\tilde{T}_1} (3,2,1)\times[2,1,6]\\
&\xrightarrow{\tilde{T}_D} (2,1)\times[3,8]
\end{aligned}
$$

Before stopped here

New Partition Identities

$\mathcal{P}(N) =$ all partitions of N.

$$
\begin{aligned}
\triangle &:= \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 > x_1 > x_n > 0\} \\
\triangle_0 &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n > x_0\} \\
\triangle_1 &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n < x_0\} \\
\triangle_D &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n = x_0\}
\end{aligned}
$$

New Partition Identities

$\tilde{T}_0$ is one-to one on $\mathcal{P}(N) \cap \triangle_0$.

$\tilde{T}_1$ is one-to one on $\mathcal{P}(N) \cap \triangle_1$.

$\tilde{T}_D$ is not one-to one on $\mathcal{P}(N) \cap \triangle_D$.

New Partition Identities

Idea:

1. Start with an interesting subset of $\mathcal{P}(N)$
2. Apply $\tilde{T}$
3. Count image

New Partition Identities

Theorem

Every number has as many integer partitions into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ *as into partitions with* $k_1 > k_m$*. Similarly, every number has as many integer partitions into partitions with* $\lambda_1 > \lambda_2 + \lambda_m$ *as into partitions with* $k_1 < k_m$*.*

New Partition Identities

$k_1 > k_m$

$\lambda_1 < \lambda_2 + \lambda_m$ (or, $m=2$, $\lambda_1 < 2\lambda_2$)

$$
\begin{array}{rcl}
(4,3)\times[1,1] & \xrightarrow{\tilde{T}_0} & (3,1)\times[2,1] \\
(3,2)\times[1,2] & \xrightarrow{\tilde{T}_0} & (2,1)\times[3,1]
\end{array}
$$

and

$k_1 < k_m$

$\lambda_1 > \lambda_2 + \lambda_m$ (or, for $m=2$, $\lambda_1 > 2\lambda_2$)

$$
\begin{array}{rcl}
(6,1)\times[1,1] & \xrightarrow{\tilde{T}_1} & (5,1)\times[1,2] \\
(5,2)\times[1,1] & \xrightarrow{\tilde{T}_1} & (3,2)\times[1,2] \\
(5,1)\times[1,2] & \xrightarrow{\tilde{T}_1} & (4,1)\times[1,3] \\
(4,2,1)\times[1,1,1] & \xrightarrow{\tilde{T}_1} & (3,2,1)\times[1,1,2] \\
(4,1)\times[1,3] & \xrightarrow{\tilde{T}_1} & (3,1)\times[1,4] \\
(3,1)\times[2,1] & \xrightarrow{\tilde{T}_1} & (2,1)\times[2,3] \\
(3,1)\times[1,4] & \xrightarrow{\tilde{T}_1} & (2,1)\times[1,5]
\end{array}
$$

New Partition Identities

With

$$\begin{aligned}
\mathcal{O} &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : \lambda_i \text{ odd}\} \\
\mathcal{F}_0 &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : \lambda_m \text{ even}, \\
&\quad \lambda_i \text{ odd if } i < m, k_1 > k_m\} \\
\mathcal{F}_1 &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : \lambda_1 \text{ even}, \\
&\quad \lambda_i \text{ odd if } i > 1, k_1 < k_m\}
\end{aligned}$$

then

$$p_{\mathcal{O}}(N) = (\text{number of odd factors of } N) + p_{\mathcal{F}_0}(N) + p_{\mathcal{F}_1}(N).$$

New Partition Identities

$$\mathcal{O} = \{(7) \times [1], (5,1) \times [1,2], (3,1) \times [2,1], (3,1) \times [1,4], (1) \times [7]\}.$$

$$\mathcal{F}_0 = \emptyset$$

$$\mathcal{F}_1 = \{(4,1) \times [1,3], (2,1) \times [2,3], (2,1) \times [1,5]\}.$$

$$\begin{aligned} p_{\mathcal{O}}(7) &= (\text{number of odd factors of } 7) + p_{\mathcal{F}_0}(7) + p_{\mathcal{F}_1}(7) \\ 5 &= 2 + 0 + 3 \end{aligned}$$

New Partition Identities

· Some of the many sets

For all m, we have $\lambda_1 > \cdots, > \lambda_m > 0$ and $k_i > 0$ for $i = 1, \ldots, m$.

upon which we found new partition identities

sets	dim $= 2$	dim ≥ 3
$\triangle_0$	$2\lambda_2 > \lambda_1$	$\lambda_2 + \lambda_m > \lambda_1$
$\triangle_1$	$2\lambda_2 < \lambda_1$	$\lambda_2 + \lambda_m < \lambda_1$
$\triangle_D$	$2\lambda_2 = \lambda_1$	$\lambda_2 + \lambda_m = \lambda_1$
$\triangle_{00}$	$2\lambda_2 > \lambda_1, 2\lambda_1 > 3\lambda_2$	$\lambda_2 + \lambda_m > \lambda_1, 2\lambda_2 < \lambda_1 + \lambda_3$
$\triangle_{01}$	$2\lambda_2 > \lambda_1, 2\lambda_1 < 3\lambda_2$	$\lambda_2 + \lambda_m > \lambda_1, 2\lambda_2 > \lambda_1 + \lambda_3$
$\triangle_{10}$	$2\lambda_2 < \lambda_1, 3\lambda_2 > \lambda_1$	$\lambda_2 + \lambda_m < \lambda_1, \lambda_2 + 2\lambda_m > \lambda_1$
$\triangle_{11}$	$3\lambda_2 < \lambda_1$	$\lambda_2 + 2\lambda_m < \lambda_1$

New Partition Identities

$M_0 = T_0(\triangle_0)$	$k_1 > k_2$	$k_1 > k_m$
$M_1 = T_1(\triangle_1)$	$k_1 < k_2$	$k_1 < k_m$
$T_0(\triangle_{00})$	$2\lambda_2 > \lambda_1, k_1 > k_2$	$\lambda_2 + \lambda_m > \lambda_1, k_1 > k_m$
$T_0(\triangle_{01})$	$2\lambda_2 < \lambda_1, k_1 > k_2$	$\lambda_2 + \lambda_m < \lambda_1, k_1 > k_m$
$T_1(\triangle_{10})$	$2\lambda_2 > \lambda_1, k_1 < k_2$	$\lambda_2 + \lambda_m > \lambda_1, k_1 < k_m$
$T_1(\triangle_{11})$	$2\lambda_2 < \lambda_1, k_1 < k_2$	$\lambda_2 + \lambda_m < \lambda_1, k_1 < k_m$
$T_0(T_0(\triangle_{00}))$	$2k_2 > k_1 > k_2$	$k_1 > k_m > k_{m-1}$
$T_1(T_0(\triangle_{01}))$	$2k_1 > k_2 > k_1$	$2k_1 > k_m > k_1$
$T_0(T_1(\triangle_{10}))$	$2k_2 < k_1$	$k_1 > k_m, k_{m-1} > k_m$
$T_1(T_1(\triangle_{11}))$	$2k_1 < k_2$	$k_m > 2k_1$

New Partition Identities

$\mathcal{D}$	$k_1 = k_2 = 1$	$k_1 = \ldots = k_m = 1$
$\mathcal{E}_0$	$k_1 = 2, k_2 = 1$	$k_1 = 2, k_2 \ldots = k_m = 1$
$\mathcal{E}_1$	$k_1 = 1, k_2 = 2$	$k_1 \ldots = k_{m-1}, k_m = 2$
$\mathcal{E}_D$	$k_1 = 2, k_2 = 2$	$k_1 = 2, k_2 \ldots = k_{m-1}, k_m = 2$
$\mathcal{O}$	λ_1, λ_2 odd	λ_i odd, $i = 1, \ldots, m$
$\mathcal{F}_0$	λ_1 odd, λ_2 even	λ_i odd $i = 1 \ldots, m-1, \lambda_2$ even
$\mathcal{F}_1$	λ_1 even, λ_2 odd	λ_1 even, λ_i odd $i = 2 \ldots, m$

Questions

There are many different multi-dimensional continued fraction algorithms.

Why use the triangle map?

Questions

$$\begin{pmatrix} M & 0 \\ 0 & (M^{-1})^T \end{pmatrix}$$

This bad

negative entries

Most multi-dimensional continued fraction algorithms seem to be not “partition friendly”.

For example, for both Mönkemeyer and Cassaigne, the multiplicities k start becoming negative numbers.

Questions

$$\begin{pmatrix} M & 0 \\ 0 & (M^{-1})^T \end{pmatrix} \geq 0$$

Recently Matthew Phang has shown that the Selmer and the Brun algorithms are partition friendly

Neither respect conjugation of the Young shape.

Neither do the other few examples that are partition friendly

Questions

Thanks

Can that the triangle map is both partition friendly and Young conjugation compatible be used to understand its dynamics?

The extended triangle map is the natural extension of the slow triangle map. Does this tell us anything?